TECHNICAL NOTE

Exact solution for entrance region flow between parallel plates

B. S. Narang*

A method of obtaining an exact solution of the complete Navier–Stokes equations for incompressible flow has been developed to study the development of fluid flow between parallel plates. An analytical solution has been developed in the form of a convergent infinite series and a numerical solution obtained for Reynolds numbers of 10 and 500. These results have been compared with data available elsewhere. The method developed here is expected to have general application

Key words: *Laminar flow, entry effects, parallel plates flow, entrance region flow*

The development of fluid flow in the entrance region of parallel plates and circular tubes is a classical problem with an extensive literature. Four methods have been used to approximate a solution of the governing Navier–Stokes equations to the entry problem: the integral method; patching of solutions; linearizing of equations of motion; and numerical methods.

The aim of this work was to develop a method of obtaining a solution of the complete Navier–Stokes equations. The solution obtained by modifying the linearized solution of Narang and Krishnamoorthy[1] reduces to the special case of the solution of the linearized differential equations.

Mathematical formulation

The Navier–Stokes equations for incompressible flow between parallel plates were linearized by Narang and Krishnamoorthy[1] and can be written in terms of stream function in dimensionless form as:

$$\frac{\partial^3\psi}{\partial x\,\partial y^2}+\frac{\partial^3\psi}{\partial x^3}=\frac{1}{N_R}\nabla^4\psi \tag{1}$$

The solution of the above equation can be written as:

$$\begin{aligned}\psi_1(x,y)=\sum_{n=1}^{N}\{&A_n\sin(1_n y)\,e^{-1_n x}\\&+B_n[\cos(\eta_n x)-\sin(\eta_n x)]\sinh(\eta_n y)\\&+C_n[\cos(\eta_n x)+\sin(\eta_n x)]\sinh(\eta_n y)\\&+D_n[\sinh(\xi_n\eta_n y)_R\cos(\eta_n x)\\&-\sinh(\xi_n\eta_n y)_I\sin(\eta_n x)]\\&+E_n[\sinh(\xi_n\eta_n y)_I\cos(\eta_n x)\\&+\sinh(\xi_n\eta_n y)_R\sin(\eta_n x)]\}\end{aligned} \tag{2}$$

* Department of Aerospace Engineering and Engineering Mechanics, San Diego State University, San Diego, Ca 92182, USA
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Notation

x	Dimensionless horizontal distance (with respect to a)
y	Dimensionless vertical distance (with respect to a)
a	Half width of channel
L	Parameter of magnitude at least equal to the entrance length
FN	$2L$
1_n	nth zero of $\cos(1_n)=\pi(2n-1)/(2)$
η_n	$n\pi/FN$
ξ_n	$(1+N_R^2/\eta_n^2)^{1/4}\,e^{i\alpha/2}$
N_R	$\rho Ua/\mu$ is the Reynolds number based on the half width
Re	Reynolds number based on channel width $=2N_R$
α	$\mathrm{Tan}^{-1}(N_R/\eta_n)$
i	$\sqrt{-1}$
N	Final integer value of the summation index. N tends to infinity
M	$\{[(5N)^2-5N]/2+5N\}/(5N)$
U	Uniform velocity entering the region between the plates
u	Dimensionless axial velocity component (with respect to U)
v	Dimensionless transverse velocity component (with respect to U)
A_n, B_n, C_n, D_n, E_n	Coefficients in Eq (2)
$a_{i,n}$, $b_{i,n}$, $c_{i,n}$, $d_{i,n}$, $e_{i,n}$	Coefficients in Eq (4)
$a'_{i,n}$, $b'_{i,n}$, $c'_{i,n}$, $d'_{i,n}$, $e'_{i,n}$	Coefficients in Eq (5)

Subscripts

R	Real value
I	Imaginary value
i,n	Summation indices

To satisfy the exact Navier–Stokes equation in the dimensionless form

$$\psi_y(\psi_{xxx}+\psi_{xyy})-\psi_x(\psi_{xxy}+\psi_{yyy})=\frac{1}{N_R}\nabla^4\psi \tag{3}$$

for the parallel plate flow problem the stream function, $\psi(x, y)$, is written as:

$$\begin{aligned}\psi=\sum_{n=1}^{N}\Bigg\{&A_n\left(\sum_{i=1}^{M}(-1)^{i-1}a_{i,n}\frac{(1_n y)^{2i-2}}{(2i-2)!}\right)\sin(1_n y)\,e^{-1_n x}\\
&+B_n\left[\left(\sum_{i=1}^{M}(-1)^{i-1}b_{i,n}\frac{(\eta_n x)^{2i-2}}{(2i-2)!}\right)\right.\\
&\left.\times(\cos(\eta_n x)-\sin(\eta_n x))\right]\sinh(\eta_n y)\\
&+C_n\left[\left(\sum_{i=1}^{M}(-1)^{i-1}c_{i,n}\frac{(\eta_n x)^{2i-2}}{(2i-2)!}\right)\right.\\
&\left.\times(\cos(\eta_n x)+\sin(\eta_n x))\right]\sinh(\eta_n y)\\
&+D_n\left[\left(\sum_{i=1}^{M}(-1)^{i-1}d_{i,n}\frac{(\eta_n x)^{2i-2}}{(2i-2)!}\right)\right.\\
&\times(\sinh(\xi_n\eta_n y)_R\cos(\eta_n x)\\
&\left.-\sinh(\xi_n\eta_n y)_I\sin(\eta_n x))\right]\\
&+E_n\left[\left(\sum_{i=1}^{M}(-1)^{i-1}e_{i,n}\frac{(\eta_n x)^{2i-2}}{(2i-2)!}\right)\right.\\
&\times(\sinh(\xi_n\eta_n y)_I\cos(\eta_n x)\\
&\left.+\sinh(\xi_n\eta_n y)_R\sin(\eta_n x)\right]\Bigg\}\end{aligned} \tag{4}$$

This stream function, ψ, can be separated into two parts, the first part representing the solution of the linearized differential equation, ψ_1, and the second part representing the modification of ψ_1 to satisfy the complete Navier-Stokes equation:

$$\begin{aligned}\psi=\psi_1+\sum_{n=1}^{N}\Bigg\{&A_n\left(\sum_{i=1}^{M}(-1)^{i-1}a'_{i,n}\frac{(1_n y)^{2i-2}}{(2i-2)!}\right)\\
&\times\sin(1_n y)\,e^{-1_n x}\\
&+B_n\left[\left(\sum_{i=1}^{M}(-1)^{i-1}b'_{i,n}\frac{(\eta_n x)^{2i-2}}{(2i-2)!}\right)\right.\\
&\left.\times(\cos(\eta_n x)-\sin(\eta_n x))\right]\sinh(\eta_n y)\\
&+C_n\left[\left(\sum_{i=1}^{M}(-1)^{i-1}c'_{i,n}\frac{(\eta_n x)^{2i-2}}{(2i-2)!}\right)\right.\\
&\left.\times(\cos(\eta_n x)+\sin(\eta_n x))\right]\sinh(\eta_n y)\\
&+D_n\left[\left(\sum_{i=1}^{M}(-1)^{i-1}d'_{i,n}\frac{(\eta_n x)^{2i-2}}{(2i-2)!}\right)\right.\\
&\times(\sinh(\xi_n\eta_n y)_R\cos(\eta_n x)\\
&\left.-\sinh(\xi_n\eta_n y)_I\sin(\eta_n x))\right]\\
&+E_n\left[\left(\sum_{i=1}^{M}(-1)^{i-1}e'_{i,n}\frac{(\eta_n x)^{2i-2}}{(2i-2)!}\right)\right.\\
&\times(\sinh(\xi_n\eta_n y)_I\cos(\eta_n x)\\
&\left.+\sinh(\xi_n\eta_n y)_R\sin(\eta_n x)\right]\Bigg\}\end{aligned} \tag{5}$$

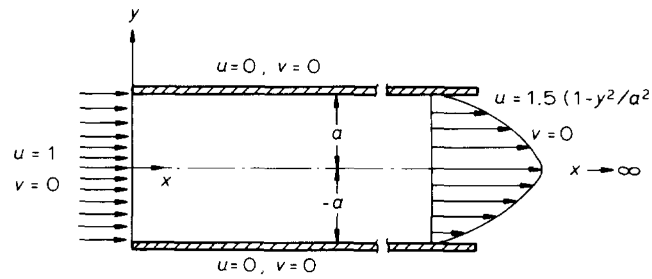


Fig 1 Boundary conditions for parallel plate flow

The form of the function, ψ, given above is not arbitrary but was arrived at only after a considerable amount of experimentation.

Numerical calculations for uniform inlet velocity

The boundary conditions for flow between parallel plates are shown in Fig. 1. The physical nature of the flow, as well as the mathematics of the problem, makes it evident that the perturbation of the flow at a point will not influence the flow at a great distance from it. If L represents the distance away from a point, within which the perturbation of the flow at a point essentially dies down, then the boundary conditions at $x=\infty$ can be replaced by

$$u=1.5-1.5y^2,\qquad v=0 \text{ at } L<x$$

The approximate interval of x for solution of the problem is then $0<x<2L$ so that the flow at $x<L$ is not influenced by the flow at $x>2L$[†].

Determination of $a_{i,n}$, $b_{i,n}$, $c_{i,n}$, $d_{i,n}$, $e_{i,n}$

In order to determine these constants, the Navier–Stokes differential equation is written as:

$$\psi_y(\psi_{xxx}+\psi_{xyy})-\psi_x(\psi_{xxy}+\psi_{yyy})=\frac{1}{N_R}(\nabla^4\psi)(\psi_y)_{x=0} \tag{6}$$

Notice that $(\psi_y)_{x=0}$ is equal to 1 for the flow problem being solved. For solution of the numerical example, only the first three terms of the series for the stream

† $L=2.5$ for $Re=10$ and $L=60$ for $Re=500$

function given by Eq (4) are chosen, that is N is set equal to three. This choice of N leads to 120 possible different nonlinear combinations of A_n^2, A_nB_n, A_nC_n, etc., of A_n, B_n, . . . , E_n when Eq (4) is substituted into Eq (6). The coefficients of the combinations A_n^2, A_nB_n, etc., which are obtained from this substitution, are set equal to zero, thus giving 120 equations. These are algebraic equations and are nonlinear in $a_{i,n}$, $b_{i,n}$, etc.

The system of 120 nonlinear equations was solved by using the subroutine developed by Powell[2]. Convergence to the final values of $a_{i,n}$, $b_{i,n}$, etc., is obtained when the sum of the squares of the residuals is zero. The Cyber 174 computing system was used for solving these equations. The sum of the squares of the residuals was reduced to 0.23 and 32.2 for Reynolds numbers of 10 and 500 respectively. It is important, however, to emphasize that the magnitude of the individual terms in most of these equations was of the order of 10^7. The largest residual was 3.6 for one equation in the case of Reynolds number of 500. Most of the equations had residuals of the order of 10^{-5} or less. It is therefore believed that the values of the coefficients $a_{i,n}$, $b_{i,n}$, $c_{i,n}$, etc., thus obtained, would be very accurate. The final values of these coefficients are given in Tables 1 and 2.

Determination of A_n, B_n, C_n, D_n, E_n

These coefficients are determined by making the stream function, ψ, in Eq (4) satisfy the boundary conditions shown in Fig. 1. The values of the coefficients are shown in Table 3.

Table 1 Values of coefficients $a_{i,n}$, $b_{i,n}$, $c_{i,n}$, $d_{i,n}$ and $e_{i,n}$ for $Re = 10$

$i=$	1	2	3	4	5	6	7	8
$a_{i,1}$	0.0125	0.0133	0.0129	0.0599	0.3283	0.3402	−0.2111	−0.0972
$a_{i,2}$	−0.0692	0.0192	−0.0209	−0.1038	−0.1380	0.1797	−0.0046	0.0888
$a_{i,3}$	0.0650	0.1152	0.3754	0.4989	0.1954	−0.0976	−0.0519	−0.0275
$b_{i,1}$	0.0129	0.0156	0.0059	0.4170	0.1254	0.1695	0.3139	0.0701
$b_{i,2}$	0.0126	0.0147	0.0200	0.0390	0.1364	0.2933	−0.2074	−0.3042
$b_{i,3}$	0.0111	0.0144	0.0244	0.0602	0.1444	0.2668	0.2475	−0.1413
$c_{i,1}$	0.0128	0.0247	0.0989	0.0530	0.1357	0.2139	0.0521	0.1270
$c_{i,2}$	0.0134	0.0138	0.0189	0.0562	0.2724	0.2248	0.0508	0.0793
$c_{i,3}$	0.0149	0.0139	0.0050	−0.0091	−0.0181	0.0102	0.1028	0.0941
$d_{i,1}$	0.0132	0.0134	0.0569	0.1107	0.3022	0.0933	0.1716	0.2880
$d_{i,2}$	0.0147	0.0134	0.0109	−0.0172	−0.1008	0.4280	−0.7346	−0.0303
$d_{i,3}$	0.0441	0.0151	0.0144	0.0274	0.0596	0.0927	0.1043	0.0904
$e_{i,1}$	0.0129	0.0258	0.1159	0.3911	0.1823	0.1756	−0.1580	−0.1472
$e_{i,2}$	0.0131	0.0142	0.0241	0.0599	0.1267	−0.0567	0.1135	0.1900
$e_{i,3}$	0.0127	0.0133	0.0231	0.0438	0.0762	0.1015	0.0999	0.0925

Table 2 Values of coefficients $a_{i,n}$, $b_{i,n}$, $c_{i,n}$, $d_{i,n}$ and $e_{i,n}$ for $Re = 500$

$i=$	1	2	3	4	5	6	7	8
$a_{i,1}$	0.0055	0.0063	0.0057	0.0277	0.2171	0.3368	−0.2409	−0.1266
$a_{i,2}$	−0.0390	−0.0771	−0.1092	−0.0414	−0.1278	0.1942	−0.0092	0.1526
$a_{i,3}$	−0.0177	−0.0307	0.4431	0.5459	0.1389	−0.1888	−0.0879	−0.0775
$b_{i,1}$	0.0060	0.0086	0.0093	0.4099	0.0733	0.1613	0.3088	0.1679
$b_{i,2}$	0.0064	0.0068	0.0085	0.0171	0.0677	0.2421	−0.1821	−0.3751
$b_{i,3}$	0.0057	0.0069	0.0145	0.0431	0.1233	0.2712	0.3098	−0.1458
$c_{i,1}$	0.0060	0.0065	0.0954	0.0718	0.1315	0.2065	0.0897	0.1249
$c_{i,2}$	0.0062	0.0064	0.0104	0.0424	0.2074	0.2247	0.0333	0.1472
$c_{i,3}$	0.0089	0.0068	0.0007	−0.0104	−0.0183	0.0082	0.1006	0.0934
$d_{i,1}$	0.0065	0.0073	0.1861	0.1087	0.2951	0.0949	0.1656	0.4424
$d_{i,2}$	0.0050	0.0070	0.0028	−0.0329	−0.0848	0.3827	−0.7745	0.0039
$d_{i,3}$	0.0240	0.0056	0.0081	0.0239	0.0572	0.0878	0.1035	0.0983
$e_{i,1}$	0.0063	0.0129	0.1844	0.3813	0.2519	0.2710	−0.0956	−0.2377
$e_{i,2}$	0.0066	0.0055	0.0145	0.0500	0.1638	−0.1050	0.0926	0.2945
$e_{i,3}$	0.0072	0.0049	0.0175	0.0426	0.0753	0.0979	0.1042	0.0904

Table 3 Values of coefficients A_n, B_n, C_n, D_n, E_n

Re	n	A_n	B_n	C_n	D_n	E_n
	1	−0.1455 E01	−0.1151 E04	−0.2456 E04	−0.3113 E01	−0.8594 E01
10	2	−0.2760 E-03	0.2099 E03	0.3058 E03	−0.2569 E-01	0.1276 E00
	3	−0.3020 E-04	−0.1484 E01	0.1529 E02	−0.2794 E-01	−0.1234 E00
	1	−0.1455 E01	−0.2760 E-03	−0.3020 E-04	−0.1151 E04	0.2099 E03
500	2	−0.1484 E01	0.2456 E04	0.3058 E03	0.1529 E02	−0.3113 E01
	3	−0.2569 E-01	−0.2794 E-01	−0.8594 E01	0.1276 E00	−0.1234 E00

Discussion

Solution of the equations for u and v, based on the first three 'N' terms of the series, was obtained for Reynolds numbers of 10 and 500. The values of the horizontal component u are plotted in Figs 2 and 3. Solid lines in these figures represent the solution of the linearized differential Eq (1) based on the first twenty terms of the series of Eq (2). The solution of linearized equation based on the first three terms of Eq (2) is also shown plotted for comparison purposes. Numerical results of Sparrow and Lin[3], Schlichting[4] and Brandt and Gillis[5], when available, are also plotted in the figures. It can be seen that the exact solution of this study, as well as the linear solution based on the first three terms, have maximum deviation closest to the entrance region when compared to the twenty term linear solution. This suggests that the first three

Fig 2 (right) Horizontal velocity profiles for various values of x *for* Re = *10*

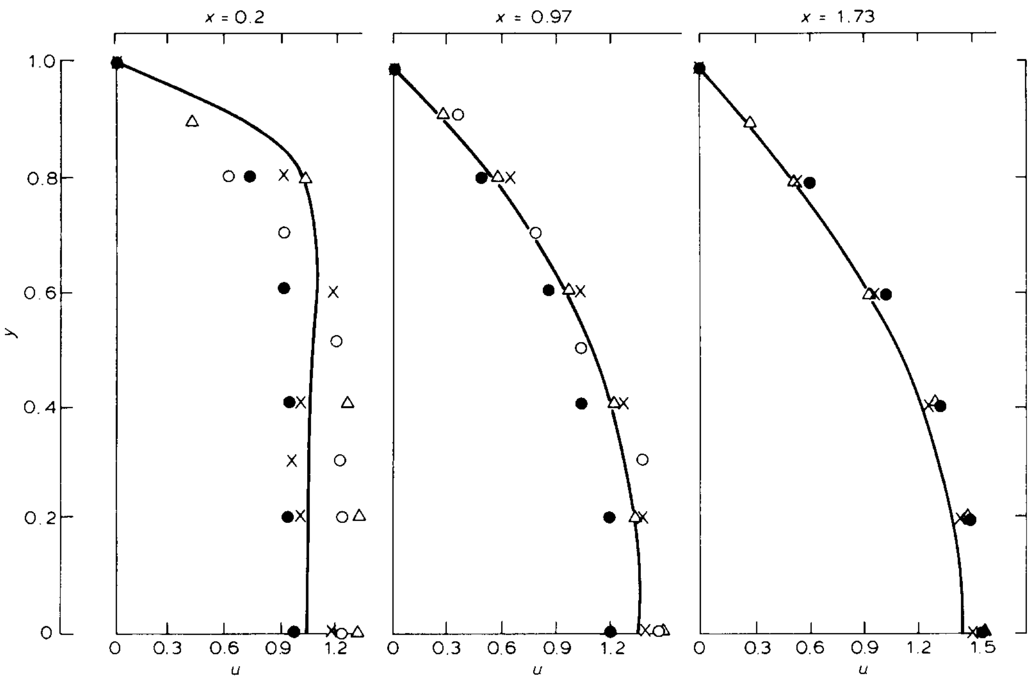
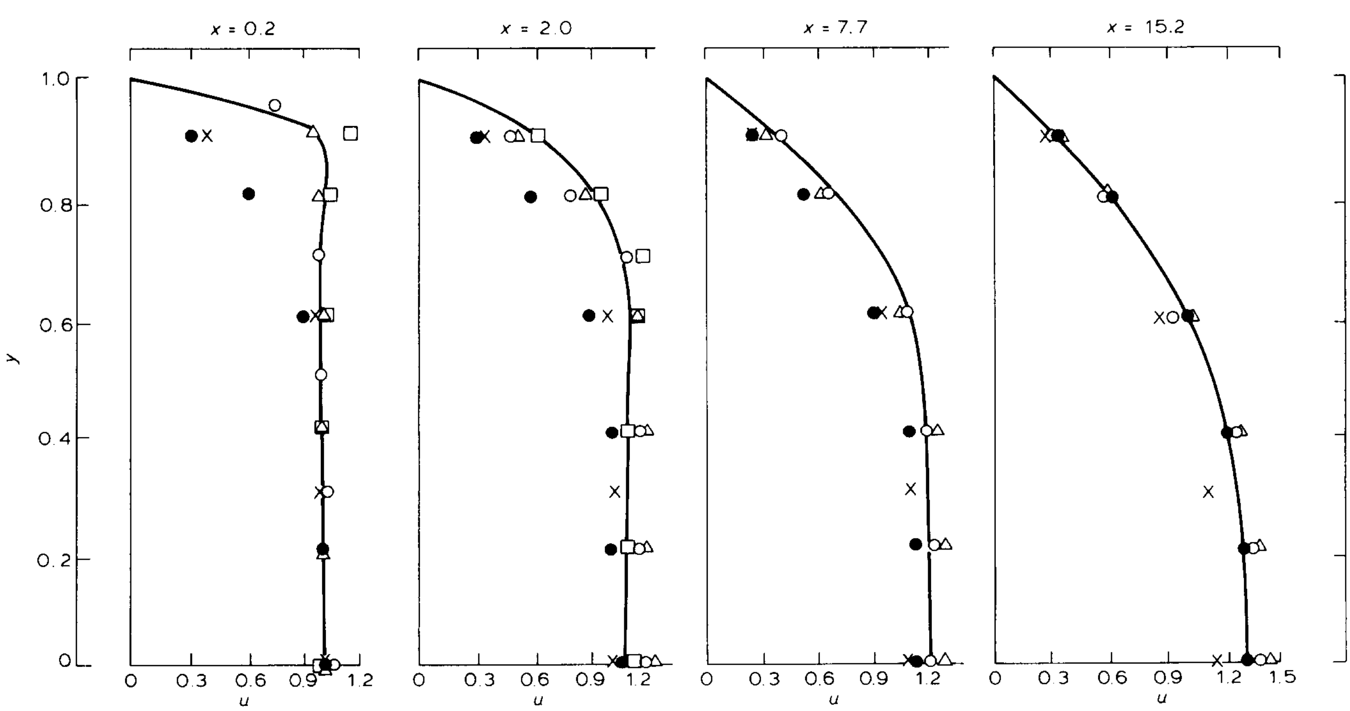


Fig 3 (below) Horizontal velocity profiles for various values of x *for* Re = *500*

—— Linearized solution based on the first 20 terms[1]
X Linearized solution based on the first 3 terms[1]
● Exact solution based on the first three terms (present investigation)
○ Sparrow and Lin[3]
△ Schlichting[4]
□ Brandt and Gillis[5]



terms of the exact solution (or that of the linear solution) are not sufficient for a good representation of the flow in the entrance region. The agreement of the exact solution gets better downstream of the entrance. It is not possible to estimate the truncation error, because the next higher value of 'N' in the series would require a computer core memory much larger than is presently available.

The application of this method can be shown to lead to the exact solution, $y = cx - c^2$, of the non-linear differential equation $y'^2 - xy' + y = 0$ with $y' = c$ (constant). The solution can be obtained by first solving the linearized equation $-xy' + y = 0$. Its solution is $y = bx$ where b is a constant of integration. Now we rewrite the original differential equation $y'^2 + (-xy' + y)(y'/c) = 0$. Next we assume that $y = A\sum_{i=1}^{n} a_i x^{i-1}$ where the constants A and a_i are to be determined. Substitution of this y in the above differential equation leads to the following coefficient of A^2:

$$\left(\sum_{i=1}^{n-1} i a_{i+1} x^{i-1}\right)^2 - (x/c)\left(\sum_{i=1}^{n-1} i a_{i+1} x^{i-1}\right)^2$$

$$+\left(\sum_{i=1}^{n} a_i x^{i-1}\right)\left(\sum_{i=1}^{n-1} i a_{i+1} x^{i-1}\right) = 0$$

This equation is satisfied for all values of x when $a_2 = -a_1/c$ and $a_3 = a_4 = \cdots = a_n = 0$. Application of the condition on y' leads to the final value of the function y.

Conclusions

A method has been developed to obtain an analytical solution, in the form of an infinite series, of the complete Navier–Stokes equations for two dimensional incompressible flow. This exact solution reduces to the linear case by setting the coefficients $a'_{i,n}$, $b'_{i,n}$, etc., in Eq (5) equal to zero. A uniform inlet velocity was assumed although the method would work for any other known inlet velocity distribution. The method developed here has been successfully applied to obtain a solution of the non-linear transonic flow equation[6]. It is hoped that other equations which are similar to the Navier–Stokes equations and the transonic flow equation can also be solved by the method developed here.

References

1. **Narang, B. S. and Krishnamoorthy, G.** Laminar flow in the entrance region of parallel plates. *Journal of Applied Mechanics, Trans. ASME, 1976, 186–188*
2. **Rabinowitz, P.** ed., Numerical Methods for Nonlinear Algebraic Equations. *Gordon and Breach, New York, 1970*
3. **Sparrow, E. M. and Lin, S. H.** Flow development in the hydrodynamic entrance region of tubes and ducts. *Physics of Fluids, 1964,* 7*(3),* 338–347
4. **Schlichting, H.** Boundary Layer Theory. *McGraw-Hill, New York, 1960*
5. **Brandt, A. and Gillis, J.** Magnetohydrodynamic flow in the inlet region of a straight channel. *Physics of Fluids, 1966,* 9*(4),* 690–699
6. **Narang, B. S.** A solution of the transonic flow equation. *Submitted for publication to Journal of Aircraft*

BOOK REVIEWS

Fundamentals of Compressible Flow

S. M. Yaha

This book has been written as an undergraduate text book on compressible fluid mechanics. In the preface the author states that there is a growth in the introduction of gas dynamics as a separate subject in engineering institutions in India and elsewhere, and so this book is aimed at such courses. It contains most of the material that might be covered at undergraduate level. After the background covered by the first three chapters, the book covers one-dimensional isentropic flow, wave motion, normal and oblique shock waves, and constant area flow with friction or heat transfer. This is followed by a chapter headed 'Multidimensional Flow' and a chapter on methods of measurement. The appendices include some gas tables (for $\gamma = 1.4$). There is, however, no mention of the method of characteristics which one might expect to see in such a book.

In the introductory chapters, the author has the difficult task of covering the background of basic thermodynamics and fluid mechanics without taking up too large a proportion of the book. As a result, one suspects that these chapters might not be particularly helpful to someone who is not already fairly familiar with the material. There are a few topics covered which do not seem directly related to later material, such as a mention of the third law of thermodynamics, the composition of the earth's atmosphere and the brief use of vector notation. Some points may irritate the thermodynamicist, such as the definition of work and the implicit assumption that isentropic flow is always adiabatic.

The main topics of compressible flow are covered quite thoroughly. Mathematically, the treatment is rigorous, with a large number of equations and their derivations presented. There is, perhaps, a lacking counter-balance in engineering appreciation of the application of the principles. For instance, in the treatment of one-dimensional flow through nozzles, there is no hint that two-dimensional effects or wall friction will modify the flow pattern. Similarly, in the treatment of constant area flow with friction (Fanno line process) or with heat transfer (Rayleigh line process) there is no discussion of the extent to which these idealised flows are likely to be realised. Quite a helpful feature for the student is the number of worked examples at the end of each chap-